THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH2058 Honours Mathematical Analysis I
Tutorial 1
Date: 13 September, 2024

TA Information:

Stephen Liu
stephenliu@cuhk.edu.hk
LSB 232A

Announcement:

- HW1 posted on course website, due 24/9 23:59pm on Gradescope
- Quiz 1 will be on 20/9 during lecture

Axioms Question Example:

For $a \neq 0$, Prove that $\frac{1}{\frac{1}{a}} = a$ e.g. A2 commutativity of addition

⋮ ⋮

At each step, label which axiom you are using

3. Let $S \subseteq \mathbb{R}$ be nonempty and u be an upper bound of S. Show that the following two statements are equivalent:

   (a) if v is any upper bound of S, then $u \le v$.

   (b) for any $\varepsilon > 0$, there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Pf: (a) $\Rightarrow$ (b): Let $\varepsilon > 0$ be given. Clearly $u - \varepsilon < u$. So by contrapositive of (a), $u - \varepsilon$ is not an upper bound of S. Since $u - \varepsilon$ is not an u.b., we can find an $s_\varepsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$.

(b) $\Rightarrow$ (a). Suppose u satisfies (b). Let v be an u.b. of S. Sps on the contrary that $v < u$. Then take $\varepsilon_0 = u - v > 0$. Then by (b), $\exists\, s_{\varepsilon_0} \in S$ s.t. $s_{\varepsilon_0} > u - \varepsilon_0 = v$. Contradicts fact that v is an u.b. of S. //

4. (Exercises 2.3.12 and 2.3.13 of [BS11])

   (a) Let $S \subseteq R$ and suppose that $s^* = \sup S$ belongs to S. If $u \notin S$, show that $\sup(S \cup \{u\}) = \sup\{s^*, u\}$.

   (b) Show that a nonempty finite set $S \subseteq \mathbb{R}$ contains its supremum.

Pf: (a): 2 cases: 1) $u < s^*$. Then clearly s^* is an u.b. of $S \cup \{u\}$. Remains to show s^* is l.u.b.
Let v be an u.b. of $S \cup \{u\}$. We have $v \geq u$, $v \geq s$ for all $s \in S$. $\Rightarrow$ $v \geq s^*$ since $s^* = \sup S$.
Then since $s^* > u$, we also have $v \geq s^* > u$.
So $s^* = \sup(S \cup \{u\})$.

2) $s^* < u$. Similar.

b) We induct on the size of the set.
Base case $\{x_1\}$. Trivially, $x_1 = \sup\{x_1\}$. ✓.
$\{x_1, x_2\}$

Induction step: Suppose $S = \{x_1, \dots, x_k\}$ and $\sup S \in S$, that is, $x_j = \sup S$ for some j in $\{1, \dots, k\}$
By previous part, we know that
$\sup\{x_1, \dots, x_k, x_{k+1}\} = \sup\{x_j, x_{k+1}\}$, and both
x_j, x_{k+1} lie in the set $\{x_1, \dots, x_{k+1}\}$.

5. By the Archimedean property of $\mathbb{R}$, we can deduce that $\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$. Prove the converse statement, i.e., assume we know that $\inf\left\{\frac{1}{n} : n \in \mathbb{N}\right\} = 0$, then prove the Archimedean property without invoking the completeness axiom.

Pf: Let $x \in \mathbb{R}$. WTS $\exists\, n_x \in \mathbb{N}$ s.t. $x \leq n_x$. If $x \leq 0$, then clearly $1 \in \mathbb{N}$ works. Now suppose $x > 0$. Then $0 < \frac{1}{x}$. Then since 0 is the infimum of the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, we know that $\frac{1}{x}$ is not a l.b. of the set.

So $\exists\, n_x \in \mathbb{N}$ s.t. $0 < \frac{1}{n_x} < \frac{1}{x}$.

Taking reciprocal, we are done. //

6. Let $r \in \mathbb{R}$ be fixed. Determine the infimum and supremum of the set $X = \{|q-r| : q \in \mathbb{Q}\}$ if they exist.

Pf: First show X is not bounded by A.P. Let $0 \leq M \in \mathbb{R}$

Then WTS $\exists\, q_M \in \mathbb{Q}$ s.t. $M < |q_M - r|$.

But by A.P. $\exists\, n_M \in \mathbb{N}$ s.t. $M + r \leq n_M$. So we are done.
($\mathbb{N} \subseteq \mathbb{Q}$.)

So $\sup X$ does not exist.

Now show $\inf X = 0$. By ε-condition of infimum, suffices to show $\forall \varepsilon > 0$, $\exists\, q_\varepsilon \in \mathbb{Q}$ s.t. $|q_\varepsilon - r| < \varepsilon$ which follows by density of $\mathbb{Q}$ in $\mathbb{R}$.